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RESEARCH REPORT No. EM-114

**The Solution of Maxwell's Equations
in Terms of a Spinor Notation. Part I:
The Initial Value Problem in Terms of
Field Strengths and the Inverse Problem**

H. E. Moses

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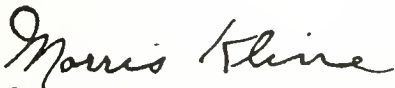
THE SOLUTION OF MAXWELL'S EQUATIONS IN TERMS OF A SPINOR NOTATION.

PART I: THE INITIAL VALUE PROBLEM IN TERMS OF FIELD STRENGTHS AND
THE INVERSE PROBLEM


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Abstract

Maxwell's equations for fields with sources in media in which the dielectric constant and permeability are unity are written in terms of a spinor notation which resembles the one used for Dirac's equation for the electron. One can introduce Green's functions and expansions in terms of complete sets of orthogonal functions, which are analagous to those used in the quantum theory of the electron, to solve Maxwell's equations in more compact form than in terms of the conventional vector notation.

It is possible to express both real and complex vector potentials simply in terms of the field strengths by means of this notation. Some examples will be given.

Furthermore, the new notation enables us to solve in a simple way an 'inverse radiation problem' which we describe as follows:

Consider at time $t < 0$ the electromagnetic field to be zero. At time $t=0$ sources are turned on and then later turned off. The electromagnetic field, which results after this process has been completed, will be a radiation field. We can solve the problem of finding the nature of the sources which will lead to a prescribed final radiation field. It is shown that, in general, the sources are not unique but additional conditions can be given which will make them so.

In Part I we discuss primarily the initial value problem and the inverse problem in terms of field strengths: vector potentials play no role whatever.

In Part II we shall introduce various vector potentials. As is well known, it is necessary to use such potentials when one wishes to discuss the interaction of charged particles with the electromagnetic field. However, in contrast to the usual procedure in which one obtains differential equations for the vector potentials and then obtains field strengths in terms of suitable derivatives, we shall do another 'inverse problem': we shall express the vector potentials in terms of the field strengths. Vector potentials so obtained will need no auxiliary conditions such as the Lorentz condition to be physically meaningful.

1. Introduction. The spinor form of Maxwell's equations

Maxwell's equations in free space with sources are:

$$(1.1) \quad \begin{aligned} \text{curl } \underline{E} + \frac{\partial}{\partial t} \underline{H} &= 0, & \text{curl } \underline{H} - \frac{\partial}{\partial t} \underline{E} &= 4\pi \underline{j}, \\ \text{div } \underline{H} &= 0, & \text{div } \underline{E} &= 4\pi \rho. \end{aligned}$$

(In (1.1) we have used Gaussian units with $c = 1$.)

As in [1] we introduce two 4-component column vectors Ψ and Φ :

$$(1.2) \quad \Psi = \begin{pmatrix} \Psi_0 \\ \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} \quad \Phi = \begin{pmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix}$$

$$(1.2a) \quad \begin{aligned} \Psi_0 &= 0 & \Phi_0 &= \rho \\ \Psi_1 &= H_1 - iE_1 & \Phi_1 &= j_1 \\ \Psi_2 &= H_2 - iE_2 & \Phi_2 &= j_2 \\ \Psi_3 &= H_3 - iE_3 & \Phi_3 &= j_3, \end{aligned}$$

$$(H_1 = H_x, H_2 = H_y, H_3 = H_z, \text{ etc.}),$$

and 4×4 matrices α^i ($i = 0, 1, 2, 3$)

$$(1.3) \quad \begin{aligned} \alpha^0 \equiv I &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \alpha^1 &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \\ \alpha^2 &= \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \\ -1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} & \alpha^3 &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Maxwell's equations have the form

$$(1.4) \quad -\frac{1}{i} \sum_{j=0}^3 \alpha^j \frac{\partial}{\partial x^j} \Psi = -4\pi\Phi$$

where, in (1.4), and later

$$(1.5) \quad x^0 = x_0 = t, \quad x^1 = -x_1 = x, \quad x^2 = -x_2 = y, \quad x^3 = -x_3 = z.$$

Equations essentially identical with (1.4) have also been given in [2] and an analogous but more cumbersome set has been given in [3]. In [4] and [5] sets of equations similar to (1.4) have been given for the truncated set of Maxwell's equations in which the divergence equations have been omitted.

In [1], the transformation properties of the vectors Ψ and Φ under the transformations of the homogeneous Lorentz group were studied, and it was shown that the usual transformation properties for the fields and sources are obtained. In particular, it was shown that the field function Ψ transformed like the wave function for a spin 1 particle.

In the present paper we shall study the solutions of (1.4) and their properties in terms of the resemblance of (1.4) to Dirac's relativistic equations for the electron. That is, we shall use techniques developed for Dirac's equations to obtain solutions of Maxwell's equations. For the most part, we shall obtain familiar results in more compact form. However, we shall also obtain some new results which have been motivated by the spinor notation. For example, we shall be able to find all time-dependent sources which lead to a prescribed final radiation field.

We shall now develop a few consequences of the representation (1.4). By explicit multiplication we note that the operators α^i ($i = 1, 2, 3$) satisfy the following multiplication laws:

$$(1.6) \quad \begin{aligned} (\alpha^1)^2 &= (\alpha^2)^2 = (\alpha^3)^2 = I \\ \alpha^1 \alpha^2 &= i \alpha^3 = - \alpha^2 \alpha^1 \\ \alpha^2 \alpha^3 &= i \alpha^1 = - \alpha^3 \alpha^2 \\ \alpha^3 \alpha^1 &= i \alpha^2 = - \alpha^1 \alpha^3 . \end{aligned}$$

Furthermore, it is useful to note that these operators are Hermitian.

As a consequence of these multiplication rules and the use of covariant and contravariant coordinates as given by (1.5), one obtains the following important identity:

$$(1.7) \quad \left(-\frac{1}{i} \sum_{j=0}^3 \alpha^j \frac{\partial}{\partial x_j} \right) \left(-\frac{1}{i} \sum_{k=0}^3 \alpha^k \frac{\partial}{\partial x^k} \right) = \left(-\frac{1}{i} \sum_{k=0}^3 \alpha^k \frac{\partial}{\partial x^k} \right) \left(-\frac{1}{i} \sum_{j=0}^3 \alpha^j \frac{\partial}{\partial x_j} \right) \\ = \square^2 \Psi = \left(\nabla^2 - \frac{\partial^2}{\partial t^2} \right) .$$

This property is very similar to a property of the Dirac operator. On applying $-\frac{1}{i} \sum_{j=0}^3 \alpha^j \frac{\partial}{\partial x_j}$ to both sides of (1.4) we obtain

$$(1.8) \quad \square^2 \Psi = \frac{4\pi}{i} \sum_{j=0}^3 \alpha^j \frac{\partial}{\partial x_j} \Phi .$$

We see immediately that when there are no source terms, the vector Ψ and hence the components of the electromagnetic field satisfy the wave equation. On the other hand, if there are sources, on taking the top component of the vectors on both sides of the equation (1.8) and using $\Psi_0 \equiv 0$, we obtain as a necessary condition for the solution of (1.4), the equation of continuity:

$$(1.9) \quad \sum_{j=0}^3 \frac{\partial \Phi_j}{\partial x^j} = \frac{\partial \rho}{\partial t} + \text{div } j = 0.$$

In most of our applications it will be useful to separate the time from the space variables and in place of (1.4) we shall write

$$(1.10) \quad \left(-\frac{1}{i} \frac{\partial}{\partial t} - H_0 \right) \Psi = -4\pi \Phi,$$

where H_0 is given by

$$(1.11) \quad H_0 = \frac{1}{i} \sum_{j=1}^3 a^j \frac{\partial}{\partial x^j}.$$

The operator H_0 is analogous to Dirac's operator for the kinetic energy of the electron. We can now state our primary objective: we propose to solve (1.10) by working essentially in the spectral representation of the operator H_0 instead of using the usual Fourier transformations in terms of wave numbers. Some use of this approach has been made in [4] and [5] but the treatment in these references is incomplete and also somewhat cumbersome because the equations analogous to (1.10), which these authors use, do not represent the complete set of Maxwell's equations.

Finally, in this section we shall define suitable inner products of column vectors and show how the energy conservation laws may be derived simply in terms of these definitions.

Let us define two different types of inner products of column vectors. Consider two vectors $A(\underline{x})$ and $B(\underline{x})$ which are given by

$$(1.12) \quad A(\underline{x}) = \begin{pmatrix} A_0(\underline{x}) \\ A_1(\underline{x}) \\ A_2(\underline{x}) \\ A_3(\underline{x}) \end{pmatrix} \quad B(\underline{x}) = \begin{pmatrix} B_0(\underline{x}) \\ B_1(\underline{x}) \\ B_2(\underline{x}) \\ B_3(\underline{x}) \end{pmatrix}.$$

In (1.12), and later, \underline{x} represents collectively the three dimensional space

coordinates (x^1, x^2, x^3) . The first type of inner product which we wish to define is essentially the Hermitian inner product in finite dimensional vector space.

We denote it by $A(\underline{x}) \cdot B(\underline{x})$

$$(1.13) \quad A(\underline{x}) \cdot B(\underline{x}) = \sum_{i=0}^3 A_i^*(\underline{x}) B_i(\underline{x}) ,$$

where the asterisk means the complex conjugate. The second inner product, which we denote by (A, B) , is the Hermitian inner product used in Hilbert space

$$(1.14) \quad (A, B) = \sum_{i=0}^3 \int A_i^*(\underline{x}) B_i(\underline{x}) d\underline{x} = \int A(\underline{x}) \cdot B(\underline{x}) d\underline{x} .$$

It is important to note that the operators α^i are Hermitian with respect to both inner products, i.e.,

$$(1.15) \quad \begin{cases} A(\underline{x}) \cdot \alpha^i B(\underline{x}) = \alpha^i A(\underline{x}) \cdot B(\underline{x}) , \\ (A, \alpha^i B) = (\alpha^i A, B) . \end{cases}$$

We shall now show how the energy density and Poynting vector may be expressed in terms of these inner products by deriving the appropriate conservation law. Let us define the vectors $A(\underline{x})$ and $B(\underline{x})$ by

$$(1.16) \quad \begin{cases} A(\underline{x}) = \frac{1}{8\pi} \mathbb{V}(\underline{x}) , \\ B(\underline{x}) = \frac{-1}{i} \sum_{j=0}^3 \alpha^j \frac{\partial}{\partial x^j} \mathbb{V} \end{cases}$$

and take the inner product (A, B) . Because of Maxwell's equations (1.4) we have

$$(1.17) \quad (A, B) = \frac{1}{8\pi} \sum_{j=0}^3 \int \mathbb{V}(\underline{x}) \cdot \alpha^j \frac{\partial}{\partial x^j} \mathbb{V}(\underline{x}) d\underline{x} = \frac{1}{2} \int \mathbb{V}(\underline{x}) \cdot \mathbb{\Phi}(\underline{x}) d\underline{x} .$$

On the other hand, on taking the product (B, A) , we see that

$$(1.18) \quad \frac{1}{8\pi} \sum_{j=0}^3 \int \alpha^j \frac{\partial}{\partial x^j} \Psi(\underline{x}) \cdot \Psi(\underline{x}) d\underline{x} = -\frac{i}{2} \int \Phi(\underline{x}) \cdot \Psi(\underline{x}) d\underline{x}$$

Since the operators α^i are Hermitian, equation (1.18) becomes

$$(1.18a) \quad \frac{1}{8\pi} \sum_{j=0}^3 \int \frac{\partial}{\partial x^j} \Psi(\underline{x}) \cdot \alpha^j \Psi(\underline{x}) d\underline{x} = -\frac{i}{2} \int \Phi(\underline{x}) \cdot \Psi(\underline{x}) d\underline{x}.$$

On adding (1.17) and (1.18a) we obtain

$$(1.19) \quad \sum_{j=0}^3 \int \frac{\partial}{\partial x^j} S^j(\underline{x}) d\underline{x} = \frac{\partial}{\partial t} \int S^0(\underline{x}) d\underline{x} + \sum_{i=1}^3 \int \frac{\partial}{\partial x^i} S^i(\underline{x}) d\underline{x} = \int P(\underline{x}) d\underline{x},$$

where

$$(1.19a) \quad \begin{cases} S^i(\underline{x}) = \Psi(\underline{x}) \cdot \alpha^i \Psi(\underline{x}) & (i = 0, 1, 2, 3) \\ P(\underline{x}) = \text{Re } i \Psi(\underline{x}) \cdot \Phi(\underline{x}) = \sum_{i=1}^3 E_i j_i \end{cases}$$

where Re means "real part."

It is easy to see that $P(\underline{x})$ is the time rate of change at which the electric field does work per unit volume on the sources. Hence S^0 is the energy density and the components S^i ($i = 1, 2, 3$) are the components of the Poynting vector. Since the field Ψ is arbitrary we can strip off the integrals in (1.19) and obtain the familiar differential form of the conservation of energy

$$(1.19b) \quad \frac{\partial S^0}{\partial t} + \text{div } \underline{S} = \underline{E} \cdot \underline{j}.$$

Various other conservation laws can be obtained in an analogous way.

2. The 'eigenfunctions' of H_0 ; the \underline{x} representation; the \underline{P} representation

It is our objective to solve the initial value problem of Maxwell's equations (1.4) in the infinite domain using techniques suggested by the resemblance of (1.4) to Dirac's equation for the electron. Accordingly, we shall work in the spectral representation of the operator H_0 . That is, we shall expand the column vectors Ψ and Φ in terms of the eigenfunctions of H_0 . In terms of

this expansion Maxwell's equations (1.4) are solved simply.

Actually, there is a slight complication in the program because of the fact that the top component of Ψ vanishes. Hence, for the purposes of expanding Ψ , we should like to distinguish the eigenfunctions of H_0 , for which the top components vanish, from the others. However, it turns out that there are only two such eigenfunctions and these do not form a complete set which span the space for which the top component of the column vector is zero. Accordingly, we introduce another 'eigenfunction' which is orthogonal to the acceptable eigenfunctions of H_0 such that the top component is zero. Then all field functions Ψ can be expanded in terms of these three 'eigenfunctions.' Finally, since we want to be able to expand vectors Φ which have top components also, we introduce a fourth 'eigenfunction' orthogonal to the other three. The last two 'eigenfunctions' are not really eigenfunctions of H_0 - only the first two are - but as indicated above they come in naturally and are actually related closely to eigenfunctions of H_0 independent of the acceptable ones.

Rather than give a detailed derivation of the 'eigenfunctions' of H_0 , we shall instead give their forms and properties.

We shall designate the four 'eigenfunctions' by $\chi(\underline{x}|\underline{p}, \epsilon)$ where \underline{p} is a three-dimensional vector and where ϵ has the values $\pm 1, 0, \gamma$. That is, the four values of ϵ label the four 'eigenfunctions.' The eigenfunctions are four component column vectors which we may also write

$$(2.1) \quad \chi(\underline{x}|\underline{p}, \epsilon) = \begin{pmatrix} \chi_0(\underline{x}|\underline{p}, \epsilon) \\ \chi_1(\underline{x}|\underline{p}, \epsilon) \\ \chi_2(\underline{x}|\underline{p}, \epsilon) \\ \chi_3(\underline{x}|\underline{p}, \epsilon) \end{pmatrix} \quad .$$

Explicitly, the set of 'eigenfunctions' which we use is

$$(2.2) \quad \left\{ \begin{array}{l} \chi(\underline{x}|\underline{p}, \epsilon) = \frac{e^{i\underline{p} \cdot \underline{x}}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2} \sqrt{\eta_x^2 + \eta_y^2}} \begin{pmatrix} 0 \\ -(\eta_x \eta_z - i\epsilon \eta_y) \\ -(\eta_y \eta_z + i\epsilon \eta_x) \\ \eta_x^2 + \eta_y^2 \end{pmatrix} \text{ for } \epsilon = \pm 1, \\ \chi(\underline{x}|\underline{p}, 0) = \frac{e^{i\underline{p} \cdot \underline{x}}}{(2\pi)^{3/2}} \begin{pmatrix} 0 \\ \eta_x \\ \eta_y \\ \eta_z \end{pmatrix}, \\ \chi(\underline{x}|\underline{p}, \epsilon) = \frac{e^{i\underline{p} \cdot \underline{x}}}{(2\pi)^{3/2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \end{array} \right.$$

In (2.2), $\underline{\eta}$ is the unit vector in the direction of \underline{p} , i.e.,

$$(2.2a) \quad \underline{\eta} = \frac{\underline{p}}{p}$$

where $p = |\underline{p}|$. Also $\underline{p} \cdot \underline{x}$ is the usual three-dimensional scalar product of \underline{p} and \underline{x} .

It is easily verified that the 'eigenfunctions' are orthonormal to each other, i.e., they satisfy

$$(2.3) \quad \sum_{\epsilon=0}^3 \int \chi_1^*(\underline{x}|\underline{p}, \epsilon) \chi_1(\underline{x}|\underline{p}', \epsilon') d\underline{x} = \int \chi(\underline{x}|\underline{p}, \epsilon) \cdot \chi(\underline{x}|\underline{p}', \epsilon') d\underline{x} = \delta(\underline{p}-\underline{p}') \delta_{\epsilon\epsilon'},$$

where

$$(2.3a) \quad \begin{array}{l} \delta_{\epsilon, \epsilon'} = 0, \quad \epsilon' \neq \epsilon \\ \delta_{\epsilon, \epsilon} = 1. \end{array}$$

This set of 'eigenfunctions' also satisfies the completeness relation:

$$(2.4) \quad \sum_{\epsilon} \int \chi_1^*(\underline{x}|\underline{p}, \epsilon) \chi_j(\underline{x}'|\underline{p}, \epsilon) d\underline{p} = \delta(\underline{x}-\underline{x}') \delta_{1j}$$

where \sum_{ϵ} means that the summation is taken over all the 'eigenfunctions.'

As a consequence we can expand any four-component column vector $A(\underline{x})$

$$A(\underline{x}) = \begin{pmatrix} A_0(\underline{x}) \\ A_1(\underline{x}) \\ A_2(\underline{x}) \\ A_3(\underline{x}) \end{pmatrix}$$

in the following way

$$(2.5) \quad A(\underline{x}) = \sum_{\underline{\epsilon}} \int \chi(\underline{x}|\underline{p}, \underline{\epsilon}) A(\underline{p}, \underline{\epsilon}) d\underline{p}$$

or

$$A_1(\underline{x}) = \sum_{\underline{\epsilon}} \int \chi_1(\underline{x}|\underline{p}, \underline{\epsilon}) A(\underline{p}, \underline{\epsilon}) d\underline{p},$$

where

$$(2.6) \quad A(\underline{p}, \underline{\epsilon}) = \sum_{\underline{i}=0}^3 \int \chi_i^*(\underline{x}|\underline{p}, \underline{\epsilon}) A_i(\underline{x}) d\underline{x} = \int \chi(\underline{x}|\underline{p}, \underline{\epsilon}) \cdot A(\underline{x}) d\underline{x}.$$

If we choose, we may regard $A_i(\underline{x})$ and $A(\underline{p}, \underline{\epsilon})$ as representing an abstract vector A in Hilbert space in two different representations, which we may call the \underline{x} -representation and the \underline{p} -representation, respectively. The relations (2.5) and (2.6) constitute a transformation between the two representations. It is easy to show that the transformation preserves the inner product in the Hilbert space sense. That is, let $A_i(\underline{x})$ and $A(\underline{p}, \underline{\epsilon})$ represent the same vector A in the \underline{x} - and \underline{p} -representations, respectively. Likewise, let $B_i(\underline{x})$ and $B(\underline{p}, \underline{\epsilon})$ represent the abstract vector B in the two representations. Then, from (2.5) and (2.6) together with the completeness and orthonormality relations (2.3) and (2.4), one can show

$$(2.7) \quad (A, B) = \sum_{\underline{i}=0}^3 \int A_i^*(\underline{x}) B_i(\underline{x}) d\underline{x} = \sum_{\underline{\epsilon}} \int A^*(\underline{p}, \underline{\epsilon}) B(\underline{p}, \underline{\epsilon}) d\underline{p}.$$

The transformation functions $\chi(\underline{x}|\underline{p}, \pm 1)$ are true eigenfunctions of H_0 which have their top components zero. They satisfy

$$(2.8) \quad H_0 \chi(\underline{x}|\underline{p}, \underline{\epsilon}) = \epsilon p \chi(\underline{x}|\underline{p}, \underline{\epsilon}), \quad (\epsilon = \pm 1), \quad \text{where } p = |\underline{p}|.$$

The 'eigenfunction' $\chi(\underline{x}|\underline{p}, 0)$ is not a true eigenfunction of H_0 . However, as shown below, H_0 acts very simply on it. This is the transformation vector which is introduced to enable us to expand all electromagnetic field functions $\Psi_i(\underline{x})$ such that $\Psi_0(\underline{x}) = 0$.

$$(2.9) \quad H_0 \chi(\underline{x}|\underline{p}, 0) = -p \chi(\underline{x}|\underline{p}, \tau), \quad (p = |\underline{p}|).$$

The transformation function $\chi(\underline{x}|\underline{p}, \tau)$ is introduced to enable us to expand any four-component column vector. The operator H_0 acts in a simple fashion on this vector also.

$$(2.10) \quad H_0 \chi(\underline{x}|\underline{p}, \tau) = -p \chi(\underline{x}|\underline{p}, 0).$$

Another relation that will prove very useful is the divergence property

$$(2.11) \quad \begin{cases} \sum_{i=1}^3 \frac{\partial}{\partial x^i} \chi_i(\underline{x}|\underline{p}, \tau) = 0, \\ \sum_{i=1}^3 \frac{\partial}{\partial x^i} \chi_i(\underline{x}|\underline{p}, 0) = \frac{ip}{(2\pi)^{3/2}} e^{ip \cdot \underline{x}}. \end{cases}$$

As we shall show, the relations (2.11) enable us to separate the longitudinal from the transverse field in a simple fashion.

We are now able to expand our time-dependent electromagnetic field vector $\Psi(\underline{x}; t)$ and sources $\Phi(\underline{x}; t)$ in terms of the 'eigenvectors' $\chi(\underline{x}|\underline{p}, \epsilon)$

$$(2.12) \quad \begin{cases} \Psi(\underline{x}; t) = \sum_{\epsilon} \int \chi(\underline{x}|\underline{p}, \epsilon) \Psi(\underline{p}, \epsilon; t) d\underline{p}, \\ \Phi(\underline{x}; t) = \sum_{\epsilon} \int \chi(\underline{x}|\underline{p}, \epsilon) \Phi(\underline{p}, \epsilon; t) d\underline{p}, \end{cases}$$

where

$$(2.12a) \quad \begin{cases} \Psi(\underline{p}, \epsilon; t) = \sum_{\underline{x}} \int \chi_i^*(\underline{x}|\underline{p}, \epsilon) \Psi_i(\underline{x}; t) d\underline{x}, \\ \Phi(\underline{p}, \epsilon; t) = \sum_{\underline{x}} \int \chi_i^*(\underline{x}|\underline{p}, \epsilon) \Phi_i(\underline{x}; t) d\underline{x}. \end{cases}$$

Since we require $\Psi_0(\underline{x};t) \equiv 0$, we see from (2.2) that in the \underline{p} -representation we must have

$$(2.13) \quad \Psi(\underline{p}, \underline{\tilde{r}}; t) = 0 .$$

Hence we may write

$$(2.14) \quad \Psi(\underline{x}; t) = \Psi^T(\underline{x}; t) + \Psi^L(\underline{x}; t) ,$$

where

$$\left\{ \begin{aligned} \Psi^T(\underline{x}; t) &= \sum_{\epsilon=\pm 1} \int \chi(\underline{x}|\underline{p}, \epsilon) \Psi(\underline{p}, \epsilon; t) d\underline{p} , \\ \Psi^L(\underline{x}; t) &= \int \lambda(\underline{x}|\underline{p}, 0) \Psi(\underline{p}, 0; t) d\underline{p} . \end{aligned} \right.$$

Now Ψ^T is the transverse field, since from (2.11) we have $\sum_{i=1}^3 \frac{\partial}{\partial x^i} \Psi_i^T = 0$, which leads to $\text{div } \underline{E}^T = \text{div } \underline{H}^T = 0$. Likewise Ψ^L is the longitudinal part of the field. From the orthogonality relations (2.3) we see that the longitudinal and transverse fields are orthogonal (in the Hilbert space sense) to each other:

$$(2.15) \quad \int \Psi^L(\underline{x}; t) \cdot \Psi^T(\underline{x}; t) d\underline{x} = 0 .$$

Because of (2.15) the energy of the field splits up into a longitudinal energy and a transverse energy: The energy E_n over all space is given by

$$(2.16) \quad E_n = \int S^0(\underline{x}) d\underline{x} = \frac{1}{8\pi} \int \Psi(\underline{x}; t) \cdot \Psi(\underline{x}; t) d\underline{x} = \frac{1}{8\pi} \int \Psi^T(\underline{x}; t) \cdot \Psi^T(\underline{x}; t) d\underline{x} + \frac{1}{8\pi} \int \Psi^L(\underline{x}; t) \cdot \Psi^L(\underline{x}; t) d\underline{x} = E_n^T + E_n^L .$$

We now consider the sources. The equation of continuity which is a necessary condition for the solution of Maxwell's equations

$$(2.17) \quad \sum_{i=0}^3 \frac{\partial \Phi_i}{\partial x^i} = 0$$

lead to, on using (2.11) and (2.12), a restriction on $\Phi(\underline{p}, \epsilon; t)$, namely

$$(2.18) \quad \Phi(\underline{p}, 0; t) = \frac{i}{p} \frac{\partial \Phi(\underline{p}, \hat{\tau}; t)}{\partial t} .$$

Hence in the \underline{p} -representation (2.18) replaces the equation of continuity.

The condition that $\Phi_1(\underline{x}; t)$ is real leads to necessary and sufficient symmetry conditions on $\Phi(\underline{p}, \epsilon; t)$, namely

$$(2.19) \quad \begin{cases} \Phi^*(\underline{p}, \pm 1; t) = \Phi(-\underline{p}, \pm 1; t) , \\ \Phi^*(\underline{p}, \hat{\tau}; t) = \Phi(-\underline{p}, \hat{\tau}; t) , \\ \Phi^*(\underline{p}, 0; t) = -\Phi(-\underline{p}, 0; t) . \end{cases}$$

We shall find (2.19) of especial importance in the 'inverse problem' of determining sources from radiation patterns.

3. The solution of the initial value problem without sources; circularly and plane polarized waves; the Green's function for the initial value problem; energy density in the \underline{p} -representation

We shall now solve the initial value problem for Maxwell's equations without sources, using expansions in terms of the 'eigenfunctions' $\chi(\underline{x}|\underline{p}, \epsilon)$. Later we shall extend our results to the cases where sources exist.

Maxwell's equations are:

$$(3.1) \quad \left(-\frac{1}{I} \frac{\partial}{\partial t} - H_0 \right) \Psi = 0 .$$

As in Section 2 we write

$$(3.2) \quad \Psi(\underline{x}; t) = \sum_{\epsilon} \int \chi(\underline{x}|\underline{p}, \epsilon) \Psi(\underline{p}, \epsilon; t) d\underline{p}$$

and substitute into (3.1). Equations (2.8) - (2.10) show us how H_0 acts on $\chi(\underline{x}|\underline{p}, \epsilon)$ and we obtain

$$(3.3) \quad -\frac{1}{i} \sum_{\varepsilon} \int \lambda(\underline{x}|\underline{p}, \varepsilon) \frac{\partial \Psi}{\partial t}(\underline{p}, \varepsilon; t) d\underline{p} - \sum_{\varepsilon=\pm 1} \int \varepsilon p \lambda(\underline{x}|\underline{p}, \varepsilon) \Psi(\underline{p}, \varepsilon; t) d\underline{p} \\ + \int p \lambda(\underline{x}|\underline{p}, \gamma) \Psi(\underline{p}, 0; t) d\underline{p} + \int p \lambda(\underline{x}|\underline{p}, 0) \Psi(\underline{p}, \gamma; t) d\underline{p} = 0.$$

Since the 'eigenfunctions' are orthonormal we may set the coefficients of $\lambda(\underline{x}|\underline{p}, \varepsilon)$ equal to zero and obtain

$$(3.4) \quad -\frac{1}{i} \frac{\partial \Psi(\underline{p}, \varepsilon; t)}{\partial t} - \varepsilon p \Psi(\underline{p}, \varepsilon; t) = 0, \quad (\varepsilon = \pm 1),$$

$$(3.4a) \quad \begin{cases} -\frac{1}{i} \frac{\partial \Psi(\underline{p}, 0; t)}{\partial t} + p \Psi(\underline{p}, \gamma; t) = 0 \\ -\frac{1}{i} \frac{\partial \Psi(\underline{p}, \gamma; t)}{\partial t} + p \Psi(\underline{p}, 0; t) = 0. \end{cases}$$

As mentioned in Section 2, the condition that $\Psi_0(\underline{x}; t) \equiv 0$ leads to

$$(3.5) \quad \Psi(\underline{p}, \gamma; t) \equiv 0$$

and hence from the second equation of (3.4a),

$$(3.6) \quad \Psi(\underline{p}, 0; t) \equiv 0.$$

As a consequence of (3.6), we see that the solutions of Maxwell's equations without sources lead to transverse fields only.

The equations (3.4) which give the transverse fields are easily solved.

The solutions are

$$(3.7) \quad \Psi(\underline{p}, \varepsilon; t) = e^{-i\varepsilon p(t-t_0)} \Psi(\underline{p}, \varepsilon; t_0), \quad (\varepsilon = \pm 1),$$

where $\Psi(\underline{p}, \varepsilon; t_0)$ is the value of $\Psi(\underline{p}, \varepsilon; t)$ at time $t=t_0$. Hence the solution of Maxwell's equations without sources is

$$(3.8) \quad \Psi(\underline{x}; t) = \sum_{\varepsilon=\pm 1} \int \lambda(\underline{x}|\underline{p}, \varepsilon) e^{-i\varepsilon p(t-t_0)} \Psi(\underline{p}, \varepsilon; t_0) d\underline{p}$$

when we give the value of $\Psi(x;t)$ at $t = t_0$ in the \underline{p} -representation. We shall shortly show how $\Psi(x;t)$ may be obtained directly in terms of $\Psi(x;t_0)$ through the use of an appropriate Green's function.

However, we shall first show that the solution (3.8) consists of a superposition of circularly polarized waves. That is to say, the solution (3.8) consists of a superposition of solutions $\Psi(x;t|\underline{p},\epsilon)$ where

$$(3.9) \quad \Psi(\underline{x};t|\underline{p},\epsilon) = \underline{\lambda}(\underline{x}|\underline{p},\epsilon)e^{-i\epsilon p t}, \quad \epsilon = \pm 1.$$

We maintain that the solutions $\Psi(\underline{x};t|\underline{p},\epsilon)$ represent circularly polarized electromagnetic waves with frequency $p = |\underline{p}|$ travelling in the direction \underline{p} for $\epsilon = 1$ and travelling in the direction $-\underline{p}$ for $\epsilon = -1$. To prove this assertion let us choose our coordinate system such that the x-axis coincides with the vector \underline{p} . Then, on using the explicit forms for $\underline{\lambda}(\underline{x}|\underline{p},\epsilon)$ given by (2.2) and noting that $\eta_x = 1$, $\eta_y = \eta_z = 0$, we have

$$(3.10) \quad \Psi(\underline{x};t|\underline{p},\epsilon) = \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2}} e^{ip(x-\epsilon t)} \begin{pmatrix} 0 \\ 0 \\ -i\epsilon \\ 1 \end{pmatrix}.$$

Since the electric field strength is the negative of the imaginary part of Ψ , while the magnetic field strength is the real part, we have

$$(3.11) \quad \begin{aligned} E_x &= 0 & H_x &= 0 \\ E_y &= \frac{\epsilon}{\sqrt{2} (2\pi)^{3/2}} \cos p(x - \epsilon t) & H_y &= \frac{-\epsilon}{\sqrt{2} (2\pi)^{3/2}} \sin p(x - \epsilon t) \\ E_z &= -\frac{1}{\sqrt{2} (2\pi)^{3/2}} \sin p(x - \epsilon t) & H_z &= \frac{1}{\sqrt{2} (2\pi)^{3/2}} \cos p(x - \epsilon t). \end{aligned}$$

These are just the usual forms for circularly polarized electromagnetic waves which travel in the positive x direction for $\epsilon = 1$ and the negative x direction for $\epsilon = -1$.

It is interesting to note that our representation leads naturally to a decomposition in terms of a sum of circularly polarized waves rather than plane polarized waves. In [4] and [5] a similar decomposition is used.

One can, of course, construct plane polarized waves from circularly polarized waves. Consider the solution of Maxwell's equations $\Psi'(\underline{x}; t | \underline{p}, \epsilon)$ defined by

$$(3.12) \quad \Psi'(\underline{x}; t | \underline{p}, \epsilon) = \frac{\Psi(\underline{x}; t | \underline{p}, \epsilon) + \epsilon \Psi(\underline{x}; t | -\underline{p}, -\epsilon)}{2}.$$

We maintain that Ψ' represents a plane polarized wave with frequency $p = |\underline{p}|$ traveling in the direction \underline{p} for $\epsilon = 1$ and in the direction $-\underline{p}$ for $\epsilon = -1$. Let us again choose the coordinate system so that the x axis coincides with the vector \underline{p} . Then by taking the real and imaginary parts of Ψ' we obtain

$$E_x = 0$$

$$H_x = 0$$

$$E_y = \frac{1}{\sqrt{2}} \frac{1}{(2\pi)^{3/2}} \frac{(\epsilon+1)}{2} \cos p(x-\epsilon t) \quad H_y = \frac{1}{\sqrt{2}} \frac{1}{(2\pi)^{3/2}} \frac{(\epsilon-1)}{2} \sin p(x-\epsilon t)$$

$$E_z = \frac{1}{\sqrt{2}} \frac{1}{(2\pi)^{3/2}} \frac{(\epsilon-1)}{2} \sin p(x-\epsilon t) \quad H_z = \frac{1}{\sqrt{2}} \frac{1}{(2\pi)^{3/2}} \frac{(\epsilon+1)}{2} \cos p(x-\epsilon t)$$

from which our assertion is proved.

We shall now express the general solution of $\Psi(\underline{x}; t)$ of Maxwell's equations in terms of the initial value $\Psi(\underline{x}; t_0)$ through the use of an appropriate Green's function. We may write the solution (3.8) as

$$(3.14) \quad \Psi_i(\underline{x}; t) = \sum_{\epsilon=\pm 1} \int \chi_i(\underline{x} | \underline{p}, \epsilon) e^{-i\epsilon p(t-t_0)} \Psi(\underline{p}, \epsilon; t_0) d\underline{p}.$$

According to (2.12a), we have

$$(3.15) \quad \Psi(\underline{p}, \epsilon; t_0) = \sum_j \int \chi_j^*(\underline{x}' | \underline{p}, \epsilon) \Psi_j(\underline{x}'; t_0) d\underline{x}'.$$

On substituting (3.15) into (3.14) we obtain

$$(3.16) \quad \Psi_i(\underline{x}; t) = \sum_{\epsilon=\pm 1} \int \int \chi_i(\underline{x}|\underline{p}, \epsilon) \chi_j^*(\underline{x}'|\underline{p}, \epsilon) e^{-i\epsilon p(t-t_0)} \Psi_j(\underline{x}'; t_0) d\underline{x}' d\underline{p}$$

which we may also write in terms of a Green's function notation as

$$(3.16a) \quad \Psi(\underline{x}; t) = \int G(\underline{x}; t | \underline{x}'; t_0) \Psi(\underline{x}'; t_0) d\underline{x}'$$

where $G(\underline{x}; t | \underline{x}'; t')$ is an operator Green's function whose elements are given by

$$(3.17) \quad G_{ij}(\underline{x}; t | \underline{x}'; t') = \sum_{\epsilon=\pm 1} \int \chi_i(\underline{x}|\underline{p}, \epsilon) \chi_j^*(\underline{x}'|\underline{p}, \epsilon) e^{-i\epsilon p(t-t')} d\underline{p}.$$

By using the explicit forms of the eigenfunctions which occur in (3.17) as given by (2.2) we can evaluate the matrix elements of the Green's function G . We find

$$(3.18) \quad \begin{cases} G_{0j}(\underline{x}, t | \underline{x}', t') = 0, & (j=0, 1, 2, 3) \\ G_{ii}(\underline{x}, t | \underline{x}', t') = \left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^i \partial x^i} \right] B(\underline{x}; t | \underline{x}'; t'), & (i=1, 2, 3) \\ G_{12}(\underline{x}, t | \underline{x}', t') = \left[-i \frac{\partial^2}{\partial t \partial x^3} + \frac{\partial^3}{\partial x^1 \partial x^2} \right] B(\underline{x}; t | \underline{x}'; t') \\ \text{and cyclically} \\ G_{ij}(\underline{x}, t | \underline{x}', t') = G_{ji}^*(\underline{x}, t | \underline{x}', t') \end{cases}$$

where

$$(3.18a) \quad B(\underline{x}; t | \underline{x}'; t') = \frac{\eta(|\underline{x} - \underline{x}'|^2 - |t - t'|^2)}{4\pi |\underline{x} - \underline{x}'|}.$$

The function $\eta(x)$ is the Heaviside step function defined by

$$(3.18b) \quad \eta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

The first of equations (3.18) follows from the fact that the top component of the eigenfunctions involved in calculating G is zero. We shall prove the second equation of (3.18) for $i = 1$. The remaining equations are proved similarly.

From (3.17) and (2.2) we have

$$\begin{aligned}
 (3.19) \quad G_{11}(\underline{x}; t | \underline{x}'; t') &= \sum_{\epsilon=\pm 1} \int \chi_1(\underline{x} | \underline{p}, \epsilon) \chi_1^*(\underline{x}' | \underline{p}, \epsilon) e^{-i\epsilon p(t-t')} d\underline{p} \\
 &= \frac{1}{2(2\pi)^3} \int \frac{e^{ip \cdot (\underline{x} - \underline{x}')}}{\eta_x^2 + \eta_y^2} \left[\eta_x^2 \eta_z^2 + \eta_y^2 \right] \left[e^{-ip(t-t')} + e^{ip(t-t')} \right] d\underline{p} \\
 &= \frac{1}{(2\pi)^3} \int \frac{e^{ip \cdot (\underline{x} - \underline{x}')}}{\eta_x^2 + \eta_y^2} \left[\eta_x^2 (1 - \eta_x^2 - \eta_y^2) + \eta_y^2 \right] \cos p(t-t') d\underline{p} \\
 &= \frac{1}{(2\pi)^3} \int e^{ip \cdot (\underline{x} - \underline{x}')} \left[1 - \eta_x^2 \right] \cos p(t-t') d\underline{p} \\
 &= \left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} \right) \frac{1}{(2\pi)^3} \int \frac{e^{ip \cdot (\underline{x} - \underline{x}')}}{p^2} \cos p(t-t') d\underline{p} .
 \end{aligned}$$

Let us now evaluate the integral. Let us set

$$(3.20) \quad \underline{\xi} = \underline{x} - \underline{x}' , \quad \xi = |\underline{\xi}| , \quad \lambda = t - t' .$$

Then, on using polar coordinates with $\underline{x} - \underline{x}'$ coinciding with the p_z axis we have

$$\begin{aligned}
 (3.21) \quad \frac{1}{(2\pi)^3} \int \frac{e^{ip \cdot (\underline{x} - \underline{x}')}}{p^2} \cos p(t-t') d\underline{p} &= \frac{1}{(2\pi)^3} \int_0^{2\pi} d\phi \int_0^\infty dp \cos p\lambda \int_0^\pi \sin \theta e^{ip\xi} \cos \theta d\theta \\
 &= \frac{2}{(2\pi)^2 \xi} \int_0^\infty \frac{dp}{p} \cos p\lambda \sin p\xi \\
 &= \frac{1}{(2\pi)^2 \xi} \int_{-\infty}^{+\infty} \frac{dp}{p} \cos p\lambda \sin p\xi .
 \end{aligned}$$

This integral can easily be evaluated by the usual methods.

One shows that if $\lambda > 0$

$$\frac{1}{(2\pi)^2 \xi} \int_{-\infty}^{+\infty} \frac{dp}{p} \cos p\lambda \sin p\xi = \frac{1}{4\pi\xi} , \quad \xi > \lambda$$

$$= 0 , \quad \xi < \lambda ,$$

while, if $\lambda < 0$,

$$\frac{1}{(2\pi)^2 \xi} \int_{-\infty}^{+\infty} \frac{dp}{p} \cos p\lambda \sin p\xi = \frac{1}{4\pi\xi} , \quad \xi > -\lambda$$

$$= 0 \quad \xi < -\lambda .$$

These results can be summarized by

$$(3.22) \quad \frac{1}{(2\pi)^2 \xi} \int_{-\infty}^{+\infty} \frac{dp}{p} \cos p\lambda \sin p\xi = \frac{1}{4\pi\xi} \eta(\xi - |\lambda|)$$

$$= \frac{1}{4\pi\xi} \eta(\xi^2 - \lambda^2)$$

$$= B(\underline{x}, t | \underline{x}', t') .$$

To conclude this section we shall discuss the \underline{p} -representation.

For Maxwell's equations without sources the \underline{p} -representation has a particularly simple interpretation, since, as a consequence of the length-preserving character of the transformation between the \underline{x} - and \underline{p} -representations, the energy of the transverse field may be written as

$$(3.23) \quad E_H^T = \sum_{\epsilon=\pm 1} \int \Psi^*(\underline{p}, \epsilon; t) \Psi(\underline{p}, \epsilon; t) d\underline{p} .$$

In the present case we have seen that the solutions of Maxwell's equations are a superposition of circularly polarized waves in various directions. Hence, from (3.23) we see that $\Psi^*(\underline{p}, \epsilon; t) \Psi(\underline{p}, \epsilon; t)$ can be interpreted as the energy density in \underline{p} space of circularly polarized radiation of frequency $p = |\underline{p}|$ propagated in the direction $\epsilon \underline{p}$.

4. The solution of the initial value problem with sources. Separation of the longitudinal and transverse fields.

We shall now use the expansion of Ψ and Φ in terms of the complete set of 'eigenfunctions' to solve Maxwell's equations with sources. As in Section 2, let us write

$$(4.1) \quad \begin{cases} \Psi(\underline{x};t) = \sum_{\epsilon} \int \chi(\underline{x}|\underline{p},\epsilon) \Psi(\underline{p},\epsilon;t) d\underline{p} , \\ \Phi(\underline{x};t) = \sum_{\epsilon} \int \chi(\underline{x}|\underline{p},\epsilon) \Phi(\underline{p},\epsilon;t) d\underline{p} . \end{cases}$$

On substituting $\Psi(\underline{x};t)$ and $\Phi(\underline{x},t)$ as given by (4.1) into Maxwell's equations

$$(4.2) \quad \left[-\frac{1}{i} \frac{\partial}{\partial t} - H_0 \right] \Psi(\underline{x};t) = \Phi(\underline{x};t) ,$$

and using (2.8)-(2.10), we obtain

$$(4.3) \quad -\frac{1}{i} \sum_{\epsilon} \int \chi(\underline{x}|\underline{p},\epsilon) \frac{\partial \Psi(\underline{p},\epsilon;t)}{\partial t} + \int \chi(\underline{x}|\underline{p},0) p \Psi(\underline{p},\tau;t) d\underline{p} \\ + \int \chi(\underline{x}|\underline{p},\tau) p \Psi(\underline{p},0;t) d\underline{p} - \sum_{\epsilon=\pm 1} \int \epsilon p \chi(\underline{x}|\underline{p},\epsilon) \Psi(\underline{p},\epsilon;t) d\underline{p} \\ = -4\pi \sum_{\epsilon} \int \chi(\underline{x}|\underline{p},\epsilon) \Phi(\underline{p},\epsilon;t) d\underline{p} .$$

By identifying coefficients of the 'eigenfunctions' $\chi(\underline{x}|\underline{p},\epsilon)$ we obtain equations for $\Psi(\underline{p},\epsilon;t)$ in terms of $\Phi(\underline{p},\epsilon;t)$, i.e., for Ψ in terms of Φ in the \underline{p} -representation. The equations for the transverse part of the field are obtained by identifying the coefficients of $\chi(\underline{x}|\underline{p},\pm 1)$ and we find

$$(4.4) \quad -\frac{1}{i} \frac{\partial}{\partial t} \Psi(\underline{p},\epsilon;t) - \epsilon p \Psi(\underline{p},\epsilon;t) = -4\pi \Phi(\underline{p},\epsilon;t) , \quad (\epsilon = \pm 1) .$$

As before, we require

$$(4.5) \quad \Psi(\underline{p},\tau;t) = 0 .$$

Hence, by identifying the coefficients of $\chi(\underline{x}|\underline{p}, \tau)$

$$(4.6) \quad \Psi(\underline{p}, 0; t) = - \frac{4\pi}{p} \Phi(\underline{p}, \tau; t),$$

and by identifying the coefficients of $\chi(\underline{x}|\underline{p}, 0)$

$$(4.7) \quad - \frac{1}{i} \frac{\partial \Psi}{\partial t}(\underline{p}, 0; t) = - 4\pi \Phi(\underline{p}, 0; t).$$

In order that $\Psi(\underline{p}, 0; t)$ be given both by (4.6) and (4.7) we obtain a relation that Φ must satisfy:

$$(4.8) \quad \frac{1}{i} \frac{1}{p} \frac{\partial \Phi(\underline{p}, \tau; t)}{\partial t} = - \Phi(\underline{p}, 0; t)$$

which is just equation (2.18): this was seen to be equivalent to the equation of continuity

$$(4.9) \quad \sum_{i=0}^3 \frac{\partial \Phi_i}{\partial x_i} = \frac{\partial \rho}{\partial t} + \text{div } \underline{j} = 0.$$

In our notation (as in the usual treatment) the transverse and longitudinal fields uncouple. The transverse field in the \underline{p} -representation is given by (4.4). Because of the first derivatives in time, one can consider an initial value problem which this poses.

On the other hand, the longitudinal field itself, rather than its derivative, depends on the sources and is therefore simpler to obtain than the transverse field. No initial value problem is involved.

Since the relation (4.6) for the longitudinal field is so simple, we shall solve it in terms of the \underline{x} -representation.

On using the expressions for the eigenfunction (2.2),

$$\begin{aligned}
 (4.10) \quad \mathbf{V}_i^L(\underline{x};t) &= -4\pi \int \frac{\chi_i(\underline{x}|\underline{p},0)}{p} \Phi(\underline{p}, \tau; t) d\underline{p} \\
 &= -4\pi \sum_j \int \frac{\chi_i(\underline{x}|\underline{p},0)}{p} \chi_j^*(\underline{x}'|\underline{p}, \tau) \Phi_j(\underline{x}'; t) d\underline{x}' \\
 &= \frac{-4\pi}{(2\pi)^3} \int \int \frac{e^{ip(\underline{x}-\underline{x}')}}{p} d\underline{p}' \eta_i \Phi_0(\underline{x}'; t) d\underline{x}', \quad i = 1, 2, 3,
 \end{aligned}$$

where $\eta_i = \eta_x$, etc. Since $\Phi_0(\underline{x}; t) = \rho(\underline{x}; t)$ we see that

$$(4.11) \quad \mathbf{V}_j^L(\underline{x}; t) = -\frac{1}{2\pi^2} \frac{\partial}{\partial x^j} \int \int \frac{e^{ip(\underline{x}-\underline{x}')}}{p^2} d\underline{p} \rho(\underline{x}'; t) d\underline{x}' .$$

But

$$\frac{1}{2\pi^2} \int \frac{e^{ip(\underline{x}-\underline{x}')}}{p^2} d\underline{p} = \frac{1}{|\underline{x}-\underline{x}'|} .$$

Hence

$$(4.12) \quad \mathbf{V}_j(\underline{x}; t) = i \frac{\partial}{\partial x^j} \int \frac{\rho(\underline{x}'; t)}{|\underline{x}-\underline{x}'|} d\underline{x}', \quad i = 1, 2, 3 .$$

$$\text{Since } \mathbf{V}_i^L = \mathbf{H}_i^L - i\mathbf{E}_i^L ,$$

we obtain the familiar result that the longitudinal magnetic field is zero.

$$(4.13) \quad \mathbf{H}^L = 0,$$

whereas the longitudinal electric field is related to charge density in the same way as the electrostatic field:

$$(4.14) \quad \mathbf{E}(\underline{x}; t) = -\text{grad} \int \frac{\rho(\underline{x}'; t)}{|\underline{x}-\underline{x}'|} d\underline{x}' .$$

For the sake of completeness we shall also write the energy of the longitudinal waves

$$(4.15) \quad E_H = \frac{1}{8\pi} \sum_i \int \mathbf{V}_i^{L*}(\underline{x}; t) \mathbf{V}_i^L(\underline{x}; t) d\underline{x} = \frac{1}{8\pi} \int \mathbf{V}^*(\underline{p}, 0; t) \mathbf{V}(\underline{p}, 0; t) d\underline{p} ,$$

which follows from the length-preserving character of the transformation from the \underline{x} to \underline{p} transformation. On using (4.6) and (2.6) and finally (4.12), then

$$\begin{aligned}
 (4.16) \quad E\tilde{n}^L &= 2\pi \int \Phi^*(\underline{p}, \tau; t) \Phi(\underline{p}, \tau; t) d\underline{p} \\
 &= \frac{2\pi}{(2\pi)^3} \int d\underline{x} \rho(\underline{x}; t) \int d\underline{x}' \rho(\underline{x}'; t) \int d\underline{p} \frac{e^{i\underline{p} \cdot (\underline{x} - \underline{x}')}}{p^2} \\
 &= \frac{1}{2} \iint \frac{d\underline{x} d\underline{x}' \rho(\underline{x}; t) \rho(\underline{x}'; t)}{|\underline{x} - \underline{x}'|}
 \end{aligned}$$

which is the usual expression for the energy of the longitudinal field.

Having disposed of the longitudinal field, we shall now discuss the initial value problem for the transverse field.

The general solution of the differential equation (4.4) is

$$(4.17) \quad \Psi(\underline{p}, \epsilon; t) = e^{-i\epsilon p(t-t_0)} \Psi(\underline{p}, \epsilon; t_0) + 4\pi i e^{-i\epsilon p t} \int_{t_0}^t \Phi(\underline{p}, \epsilon; t') e^{i\epsilon p t'} dt', \quad (\epsilon = \pm 1).$$

The first term on the right represents a solution of Maxwell's equation without sources, while the second term shows the effect of sources. It should be noted that $\Phi_0(\underline{x}', t) = \rho(\underline{x}; t)$ has no effect whatsoever on the transverse field.

We can also write the solution (4.17) in terms of the \underline{x} -representation.

We have, as a consequence of (2.12) and (2.12a),

$$\begin{aligned}
 (4.18) \quad \Psi_i^T(\underline{x}; t) &= \sum_{\epsilon=\pm 1} \int \chi_i(\underline{x}|\underline{p}, \epsilon) \Psi(\underline{p}, \epsilon; t) d\underline{p} \\
 &= \sum_{\epsilon=\pm 1} \iint \chi_i(\underline{x}|\underline{p}, \epsilon) \chi_j^*(\underline{x}'|\underline{p}, \epsilon) e^{-i\epsilon p(t-t_0)} \Psi_j(\underline{x}', t_0) d\underline{x}' d\underline{p} \\
 &\quad + 4\pi i \int_{t_0}^t dt' \iint \sum_{\epsilon=\pm 1} \chi_i(\underline{x}|\underline{p}, \epsilon) \chi_j^*(\underline{x}'|\underline{p}, \epsilon) e^{-i\epsilon p(t-t')} \Phi_j(\underline{x}'; t') d\underline{x}' d\underline{p}.
 \end{aligned}$$

Hence

$$(4.19) \quad \mathbf{V}^T(\underline{x};t) = \int d\underline{x}' G(\underline{x};t|\underline{x}';t_0) \mathbf{V}^T(\underline{x}';t_0) + 4\pi i \int_{t_0}^t dt' \int d\underline{x}' G(\underline{x};t|\underline{x}';t') \mathbf{\Phi}(\underline{x}';t')$$

where $G(\underline{x};t|\underline{x}';t')$ is the Green's function given by (3.17) and (3.18). Equations (4.19) for \mathbf{V}^T and (4.12) for \mathbf{V}^L represent the complete solution of the initial value problem for Maxwell's equations with sources.

5. The influence on radiation of sources which are switched on for a finite time. The problem of finding sources which will give a prescribed radiation field.

In the previous section we have solved the problem of obtaining the electromagnetic field in terms of the sources in the time-dependent formulation. In the present section we want to consider a special situation problem, namely the case in which the sources are identically zero except for a finite time interval. Such a situation leads to a special initial value problem.

Before the sources are switched on, the only solutions of Maxwell's equations are those given in Section 3 which correspond to electromagnetic radiation. In the interval during which the sources are on, one obtains a more complicated transverse field and also a longitudinal field. After the sources are switched off, one again obtains solutions corresponding to electromagnetic radiation, which, however, differ in general from the initial fields before the sources were switched on.

A physically interesting problem is to obtain the final field from the initial field and sources. The problem is analagous to the scattering problem in quantum mechanics. Since we have given the general solution of Maxwell's equations in the previous section, we can easily obtain the final field from the initial field and the sources.

We shall also be able to solve the 'inverse' problem which may be

described in the following way: we prescribe the initial field and the final field. We are required to find the sources which lead from the initial field to the final field. A particular case is that which occurs when the initial field is zero. We shall then want to find the sources which put energy into a prescribed radiation pattern.

Let us now assume that there are no sources for $t < t_0$. At $t = t_0$ the sources are switched on, permitted to vary in time in any desired fashion, and then switched off again at $t \geq t_1$. From (4.17) it is clear that for $t < t_0$ we have a solution of Maxwell's equations without sources which is a purely transverse field which we may write as

$$(5.1) \quad \mathbf{V}(\underline{x};t) = \mathbf{V}^T(\underline{x};t) = \sum_{\epsilon=\pm 1} \int \chi(\underline{x}|\underline{p},\epsilon) e^{-i\epsilon p(t-t_0)} \mathbf{V}(\underline{p},\epsilon;t_0) d\underline{p} \\ = \int G(\underline{x};t|\underline{x}',t_0) \mathbf{V}^T(\underline{x}';t_0) d\underline{x}', \quad (t \leq t_0).$$

In the time interval $t_0 \leq t \leq t_1$, the transverse field is given by (4.17) or (4.19).

The longitudinal field is given by (4.12).

For times $t \geq t_1$ corresponding to the switching off again of the sources we have another solution of Maxwell's equations without sources which is a pure transverse field:

$$(5.2) \quad \mathbf{V}(\underline{x};t) = \mathbf{V}^T(\underline{x};t) = \sum_{\epsilon=\pm 1} \int \chi(\underline{x}|\underline{p},\epsilon) e^{-i\epsilon p(t-t_1)} \mathbf{V}(\underline{p},\epsilon;t_1) \\ = \int G(\underline{x};t|\underline{x}',t_1) \mathbf{V}^T(\underline{x}';t_1) d\underline{x}', \quad t \geq t_1$$

where

$$(5.3) \quad \mathbf{V}(\underline{p},\epsilon;t_1) = e^{-i\epsilon p(t_1-t_0)} \mathbf{V}(\underline{p},\epsilon;t_0) + 4\pi i e^{-i\epsilon p t_1} \int_{t_0}^{t_1} \Phi(\underline{p},\epsilon;t') e^{i\epsilon p t'} dt', \quad \epsilon=\pm 1.$$

It will be useful to introduce the notation

$$(5.4) \quad F(\underline{p}, \epsilon) = \frac{1}{\sqrt{2\pi}} \int_{t_0}^{t_1} \underline{\Phi}(\underline{p}, \epsilon; t') e^{i\epsilon p t'} dt' = \frac{1}{\sqrt{2\pi}} \int_{t_0}^{t_1} \int \chi(\underline{x} | \underline{p}, \epsilon) \cdot \underline{\Phi}(\underline{x}; t') e^{i\epsilon p t'} d\underline{x} dt',$$

($\epsilon = \pm 1$) .

Equation (5.3) can be written

$$(5.5) \quad \Psi(\underline{p}, \epsilon; t_1) = e^{-i\epsilon p(t_1 - t_0)} \Psi(\underline{p}, \epsilon; t_0) + 2(2\pi)^{3/2} i e^{-i\epsilon p t_1} F(\underline{p}, \epsilon), \quad (\epsilon = \pm 1) .$$

In terms of the \underline{p} -representation, the initial electromagnetic field is given by $\Psi(\underline{p}, \epsilon; t_0)$, while the final one is given by $\Psi(\underline{p}, \epsilon; t_1)$. If we are given the sources $\underline{\Phi}(\underline{x}; t)$, we can easily calculate the final field from the initial field and sources by using (5.5) and (5.4).

Let us now consider the inverse problem. In this case we are given the initial and final fields, $\Psi(\underline{p}, \epsilon; t_0)$ and $\Psi(\underline{p}, \epsilon; t_1)$ respectively, and we are required to find the sources $\underline{\Phi}(\underline{x}; t)$. We shall show that the solution is not unique, but that we can obtain essentially unique results by imposing additional conditions.

From (5.5) we can find $F(\underline{p}, \epsilon)$ from the initial and final fields:

$$(5.6) \quad F(\underline{p}, \epsilon) = \frac{-i}{2(2\pi)^{3/2}} \left[e^{i\epsilon p t_1} \Psi(\underline{p}, \epsilon; t_1) - e^{i\epsilon p t_0} \Psi(\underline{p}, \epsilon; t_0) \right]$$

where $\epsilon = \pm 1$.

If we can find $\underline{\Phi}(\underline{p}, \epsilon; t)$ and hence $\underline{\Phi}(\underline{x}; t)$ from $F(\underline{p}, \epsilon)$, our problem is solved.

Let us define the function $F(\underline{p}, \epsilon; k)$ as being the Fourier transform of $\underline{\Phi}(\underline{p}, \epsilon; t)$ with respect to time, for $\epsilon = \pm 1$.

That is

$$(5.7) \quad \begin{cases} F(\underline{p}, \epsilon; k) = \frac{1}{\sqrt{2\pi}} \int_{t_0}^{t_1} \Phi(\underline{p}, \epsilon; t) e^{ikt} dt \\ \Phi(\underline{p}, \epsilon; t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(\underline{p}, \epsilon; k) e^{-ikt} dk \end{cases} .$$

The condition that $\Phi(\underline{x}; t)$ be real leads to equations (2.19) and hence also

$$(5.7a) \quad F^*(\underline{p}, \epsilon; k) = F(-\underline{p}, \epsilon; -k) .$$

It is clear that if we are given the initial and final fields and hence $F(\underline{p}, \epsilon)$, we can take any function $F(\underline{p}, \epsilon; k)$ which satisfies (5.7a) such that

$$(5.8) \quad F(\underline{p}, \epsilon; \epsilon p) = F(\underline{p}, \epsilon) ,$$

and obtain $\Phi(\underline{p}, \epsilon; t)$ for $\epsilon = \pm 1$ using the second of equations (5.7).

We can find a suitable real source $\Phi(\underline{x}; t)$ from

$$(5.9) \quad \Phi(\underline{x}; t) = \sum_{\epsilon} \int \chi(\underline{x} | \underline{p}, \epsilon) \Phi(\underline{p}, \epsilon; t) d\underline{p} ,$$

where $\Phi(\underline{p}, \pm 1; t)$ is given by (5.7), $\Phi(\underline{p}, \tilde{\tau}; t)$ is any arbitrary function which satisfies (2.19) and which vanishes identically when $t < t_0$ or $t > t_1$, and $\Phi(\underline{p}, 0; t) = \frac{i}{p} \frac{\partial \Phi(\underline{p}, \tilde{\tau}; t)}{\partial t} .$

We note that there are two types of lack of uniqueness in $\Phi(\underline{x}; t)$.

One type consists in the arbitrary choice of function $\Phi(\underline{p}, \tilde{\tau}; t)$ which satisfies (2.19), namely $\Phi^*(\underline{p}, \tilde{\tau}; t) = \Phi(-\underline{p}, \tilde{\tau}; t)$, but vanishes identically for times $t < t_0$ and for times $t > t_1$. This arbitrariness corresponds to the possibility of having any arbitrary charge distribution $\rho(\underline{x}; t)$ for $t_0 \leq t \leq t_1$ which would give rise to the longitudinal field given by (4.12) .

$$\mathbf{v}^L = \frac{1}{c} \frac{\partial}{\partial x^0} \int \frac{\rho(\underline{x}'; t)}{|\underline{x} - \underline{x}'|} d\underline{x}' .$$

When this charge distribution is switched off, it will not affect the final transverse field in any way whatever. Thus the final radiation field is given only by the sources

$$\Phi(\underline{x}; t) = \sum_{\epsilon=\pm 1} \int \chi(\underline{x}|\underline{p}, \epsilon) \Phi(\underline{p}, \epsilon; t) d\underline{p}$$

only. To this source we may add Φ^{Arb} where

$$\Phi^{\text{Arb}}(\underline{x}; t) = \int \chi(\underline{x}|\underline{p}; t) \Phi(\underline{p}, \tau; t) d\underline{p} + \int \chi(\underline{x}|\underline{p}, 0) \Phi(\underline{p}, 0; t) d\underline{p}$$

which, when expressed in terms of the arbitrary charge density ρ is

$$\begin{aligned} \Phi_0^{\text{Arb}}(\underline{x}; t) &= \rho(\underline{x}; t) \\ (5.10) \quad \Phi_i^{\text{Arb}}(\underline{x}; t) &= \frac{\partial}{\partial x^i} \frac{1}{4\pi} \int \frac{1}{|\underline{x} - \underline{x}'|} \frac{\partial \rho(\underline{x}'; t)}{\partial t} d\underline{x}' , \quad (i=1,2,3) . \end{aligned}$$

It is clear that Φ^{Arb} satisfies the equation of continuity $\sum_{i=0}^3 \frac{\partial \Phi_i^{\text{Arb}}}{\partial x^i} = 0$, since $\nabla^2 \frac{1}{4\pi} \frac{1}{|\underline{x}|} = -\delta(\underline{x})$.

The choice of $\rho(\underline{x}; t)$ and hence also Φ^{Arb} is not an essential lack of uniqueness in the inverse problem because this source does not give rise to transverse fields at any time: these sources give rise only to the longitudinal fields which have no propagation characteristics.

The second lack of uniqueness in the inverse problem is far more important. It concerns the choice of function $F(\underline{p}, \epsilon; k)$ for $\epsilon=\pm 1$. Except for the requirement that this function satisfy (5.7a) and (5.8), it is arbitrary. We shall therefore impose additional conditions which will make the problem unique.

6. The statement of the inverse problem which leads to unique solutions. Examples.

It will be convenient to choose the origin of time so that $t_0 = -T$ and $t_1 = T$ where $T > 0$. This can always be done by taking $T = \frac{1}{2}(t_1 - t_0)$ and introducing a new time coordinate $t' = t - \frac{t_1 + t_0}{2}$. We shall assume the new time coordinate is always used in what follows and drop the prime.

We shall prescribe the initial and final fields in the \underline{p} -representation which are now $\Psi(\underline{p}, \varepsilon; -T)$ and $\Psi(\underline{p}, \varepsilon; T)$, respectively. Henceforth, whenever ε appears it will be restricted to the values $\varepsilon = \pm 1$.

The Ansatz which will lead to unique sources for any choice of initial and final fields is the requirement that the source vector $\Phi(\underline{x}; t)$ be represented by

$$(6.1) \quad \Phi(\underline{x}; t) = \Phi^E(\underline{x}) h^E(t) + \Phi^U(\underline{x}) h^U(t),$$

where $h^E(t)$ is a prescribed real even function of t and $h^U(t)$ is a prescribed real odd function of t (the superscript U stands for 'uneven' and is used instead of O to prevent confusion). Neither $h^E(t)$ nor $h^U(t)$ is allowed to be identically zero. Our statement of the inverse problem is that we shall be able to find unique real 4-component column vectors $\Phi^E(\underline{x})$ and $\Phi^U(\underline{x})$ which are functions of \underline{x} only for any initial and final fields.

It will also be shown that the sources $\Phi(\underline{x}; t)$ so obtained will be independent of the normalization of $h^E(t)$ and $h^U(t)$ in the sense that if these functions replaced by $Ah^E(t)$ and $Bh^U(t)$ respectively, where A and B are any real, non-zero constants, the same sources $\Phi(\underline{x}; t)$ will result.

Let us first introduce the Fourier transforms of $h^E(t)$ and $h^U(t)$:

$$(6.2) \quad \begin{cases} g^E(k) = \frac{1}{\sqrt{2\pi}} \int_{-T}^{+T} h^E(t) e^{ikt} dt, \\ g^U(k) = \frac{1}{\sqrt{2\pi}} \int_{-T}^{+T} h^U(t) e^{ikt} dt. \end{cases}$$

It is easily seen that $g^E(k)$ is a real even function of k and that $g^U(k)$ is a purely imaginary odd function of k .

$$(6.3) \quad \begin{cases} g^E(k)^* = g^E(k) \\ g^E(-k) = g^E(k) \\ g^U(k)^* = -g^U(k) \\ g^U(-k) = -g^U(k) . \end{cases}$$

From (5.7) and the relation

$$(6.4) \quad \Phi(\underline{p}, \varepsilon; t) = \Phi^E(\underline{p}, \varepsilon) h^E(t) + \Phi^U(\underline{p}, \varepsilon) h^U(t)$$

we see that

$$(6.5) \quad F(\underline{p}, \varepsilon; k) = \Phi^E(\underline{p}, \varepsilon) g^E(k) + \Phi^U(\underline{p}, \varepsilon) g^U(k) ,$$

where

$$(6.6) \quad \begin{cases} \Phi^{E,U}(\underline{p}, \varepsilon) = \sum_i \int \chi_i^*(\underline{x}|\underline{p}, \varepsilon) \Phi_i^{E,U}(\underline{x}) d\underline{x} \\ \Phi_i^{E,U}(\underline{x}) = \sum_{\varepsilon=\pm 1} \int \chi(\underline{x}|\underline{p}, \varepsilon) \Phi^{E,U}(\underline{p}, \varepsilon) d\underline{p} . \end{cases}$$

The reality conditions on $\Phi^E(\underline{x})$ and $\Phi^U(\underline{x})$ lead to the requirement that

$$(6.7) \quad \Phi^{E,U}(-\underline{p}, \varepsilon) = \Phi^{E,U*}(\underline{p}, \varepsilon) .$$

Equation (5.8), together with (6.3) yields, finally,

$$(6.8) \quad F(\underline{p}, \varepsilon) = \Phi^E(\underline{p}, \varepsilon) g^E(\underline{p}) + \varepsilon \Phi^U(\underline{p}, \varepsilon) g^U(\underline{p}) .$$

Now it will be convenient to introduce $F^E(\underline{p}, \varepsilon)$ and $F^U(\underline{p}, \varepsilon)$, defined by

$$(6.9) \quad \begin{cases} F^E(\underline{p}, \varepsilon) = \frac{1}{2} [F(\underline{p}, \varepsilon) + F(-\underline{p}, \varepsilon)^*] \\ F^U(\underline{p}, \varepsilon) = \frac{1}{2} [F(\underline{p}, \varepsilon) - F(-\underline{p}, \varepsilon)^*] . \end{cases}$$

It is clear that

$$(6.10) \quad \begin{cases} F(\underline{p}, \epsilon) = F^E(\underline{p}, \epsilon) + F^U(\underline{p}, \epsilon) \\ F^E(-\underline{p}, \epsilon) = F^E(\underline{p}, \epsilon)^* \\ F^U(-\underline{p}, \epsilon) = -F^U(\underline{p}, \epsilon)^* . \end{cases}$$

Therefore, using (6.3) and (6.7), we have

$$(6.11) \quad \begin{cases} F^E(\underline{p}, \epsilon) = \Phi^E(\underline{p}, \epsilon) g^E(p) \\ F^U(\underline{p}, \epsilon) = \epsilon \Phi^U(\underline{p}, \epsilon) g^U(p) . \end{cases}$$

Furthermore, on using (5.6), we find the following solutions for $\Phi^{E,U}(\underline{p}, \epsilon)$ in terms of the initial and final fields:

$$(6.12) \quad \Phi^E(\underline{p}, \epsilon) = \frac{-i}{4(2\pi)^{3/2} g^E(p)} \left[e^{i\epsilon p T} \Psi(\underline{p}, \epsilon; T) - e^{-i\epsilon p T} \Psi(-\underline{p}, \epsilon; T)^* - e^{-i\epsilon p T} \Psi(\underline{p}, \epsilon; -T) + e^{i\epsilon p T} \Psi(-\underline{p}, \epsilon; -T)^* \right] .$$

$$(6.12a) \quad \Phi^U(\underline{p}, \epsilon) = \frac{-i\epsilon}{4(2\pi)^{3/2} g^U(p)} \left[e^{i\epsilon p T} \Psi(\underline{p}, \epsilon; T) + e^{-i\epsilon p T} \Psi(\underline{p}, \epsilon; -T)^* - e^{-i\epsilon p T} \Psi(\underline{p}, \epsilon; -T) - e^{i\epsilon p T} \Psi(-\underline{p}, \epsilon; T)^* \right] .$$

Finally, we have the sources

$$(6.13) \quad \Phi(\underline{x}; t) = \sum_{\epsilon=\pm 1} \int \chi(\underline{x}|\underline{p}, \epsilon) \Phi^E(\underline{p}, \epsilon) d\underline{p} h^E(t) + \sum_{\epsilon=\pm 1} \int \chi(\underline{x}|\underline{p}, \epsilon) \Phi^U(\underline{p}, \epsilon) d\underline{p} h^U(t) .$$

We see that the presence of the ratios $\frac{h^E(t)}{g^E(p)}$, $\frac{h^U(t)}{g^U(p)}$ in (6.13) when (6.12) and (6.12a) are substituted for $\Phi^{E,U}(\underline{p}, \epsilon)$ leads us to our conclusion that $\Phi(\underline{x}; t)$ is independent of the normalizations of $h^E(t)$ and $h^U(t)$ in the sense mentioned before,

To summarize: we prescribe the initial field $\Psi(\underline{p}, \epsilon; -T)$ and the final

field $\Psi(\underline{p}, \epsilon; T)$. We require $\Phi(\underline{x}; t)$ to have the form (6.1) where $h^E(t)$ and $h^U(t)$ are given real even and odd functions respectively of t . We then compute $\Phi^E(\underline{p}, \epsilon)$ and $\Phi^U(\underline{p}, \epsilon)$ using (6.12) and (6.12a) and finally $\Phi(\underline{x}; t)$ from (6.13).

It should be noted that it is only for exceptional cases of initial and final fields that either $\Phi^E(\underline{x})$ or $\Phi^U(\underline{x})$ is identically zero or that $\Phi^E(\underline{x}) = \Phi^U(\underline{x})$. These situations correspond to the cases in which the sources can be written $\Phi^E(\underline{x})h^E(t)$, $\Phi^U(\underline{x})h^U(t)$, or $\Phi(\underline{x})h(t)$ where $h(t)$ is an arbitrary given function of t .

We shall now consider two examples of the inverse problem.

Example 1. We shall take the field before the sources are switched on to be zero. After the sources are switched off the field is to consist of a circularly polarized wave travelling in the positive x -direction with frequency ν .

Hence

$$(6.14) \quad \begin{cases} \Psi(\underline{p}, \epsilon; -T) = 0 \\ \Psi(\underline{p}, \epsilon; T) = K \delta(\underline{p}_x - \nu) \delta(\underline{p}_y) \delta(\underline{p}_z) \delta_{\epsilon, +1}, \quad \nu > 0. \end{cases}$$

Furthermore, we shall take

$$(6.15) \quad \begin{cases} h^E(t) = A \delta(t) \\ h^U(t) = B \delta'(t). \end{cases}$$

All sufficiently short time $h^E(t)$ and $h^U(t)$ functions can be approximated by the functions given in (6.15). Furthermore, we may take T to be arbitrarily small. In fact we shall take T to be zero after the various integrations over time have been performed.

Then, in terms of the \underline{x} -representation, the field is identically zero for $t < 0$ and for $t > 0$ is given by (see 3.11)

$$\begin{aligned}
 (6.16) \quad E_x &= 0, & H_x &= 0 \\
 E_y &= \frac{K}{\sqrt{2} (2\pi)^{3/2}} \cos v(x-t), & H_y &= \frac{-K}{\sqrt{2} (2\pi)^{3/2}} \sin v(x-t), \\
 E_z &= \frac{-K}{\sqrt{2} (2\pi)^{3/2}} \sin v(x-t), & H_z &= \frac{K}{\sqrt{2} (2\pi)^{3/2}} \cos v(x-t).
 \end{aligned}$$

The calculations for the sources are quite straightforward and one obtains

$$(6.17) \quad \Phi^E(\underline{x}) = \frac{K}{A 16 \pi^{5/2}} \begin{pmatrix} 0 \\ 0 \\ -\cos vx \\ \sin vx \end{pmatrix}, \quad \Phi^H(\underline{x}) = \frac{K}{B 16 \pi^{5/2}} \begin{pmatrix} 0 \\ 0 \\ \sin vx \\ \cos vx \end{pmatrix}$$

and

$$(6.18) \quad \Phi(\underline{x};t) = \frac{K}{16 \pi^{5/2}} \begin{pmatrix} 0 \\ 0 \\ -\cos vx \delta(t) + \frac{\sin vx}{v} \delta'(t) \\ \sin vx \delta(t) + \frac{\cos vx}{v} \delta'(t) \end{pmatrix}.$$

One can add to this source the arbitrary charge distribution $\rho(\underline{x};t)$ which we may choose, if we wish, to take the form

$$\rho(\underline{x};t) = \rho(\underline{x})\delta(t)$$

where $\rho(\underline{x})$ is arbitrary. This gives rise to the arbitrary additional sources

$$(6.19) \quad \Phi^{Arb}(\underline{x};t) = \begin{pmatrix} \rho(\underline{x})\delta(t) \\ \frac{1}{4\pi} \frac{\partial}{\partial x} \int \frac{\rho(\underline{x}')}{|\underline{x}-\underline{x}'|} d\underline{x}' \delta'(t) \\ \frac{1}{4\pi} \frac{\partial}{\partial y} \int \frac{\rho(\underline{x}')}{|\underline{x}-\underline{x}'|} d\underline{x}' \delta'(t) \\ \frac{1}{4\pi} \frac{\partial}{\partial z} \int \frac{\rho(\underline{x}')}{|\underline{x}-\underline{x}'|} d\underline{x}' \delta'(t) \end{pmatrix}.$$

Having obtained the sources, we can now obtain the fields for all time using techniques given in Section 4. The transverse field is given by

$$(6.20) \quad \mathbf{V}^T(\underline{x};t) = \frac{K}{\sqrt{2} (2\pi)^{3/2}} \left\{ \eta(t) e^{i\nu(\underline{x}-t)} \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} + \frac{i\delta(t)}{\nu} \begin{pmatrix} 0 \\ 0 \\ \sin \nu x \\ \cos \nu x \end{pmatrix} \right\},$$

where $\eta(t)$ is the Heaviside function

$$(6.20a) \quad \eta(t) = 0 \quad t < 0 \\ = 1 \quad t > 0.$$

The longitudinal field is

$$(6.21) \quad \mathbf{V}^L(\underline{x};t) = i \begin{pmatrix} 0 \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \int \frac{\rho(\underline{x}')}{|\underline{x}-\underline{x}'|} d\underline{x}' \delta(t).$$

Example 2.

In the previous example the final field filled all of space shortly after the sources were switched off. As a consequence it is seen that the sources also fill all of space during the time that they are on.

In the present example we shall study the case in which the time dependence of the sources is the same as before and where we again require the initial field to be identically zero. In contrast to the previous example, however, we shall require the final transverse field to be highly concentrated in space immediately after the sources are switched off. We shall then see that as long as the sources are on, they will also be highly concentrated in space.

It would be nice to consider the field immediately after the switch-off to be

$$(6.22) \quad \mathbf{V}^A(\underline{x}) = R\delta(\underline{x})$$

where R is the four-component vector

$$(6.22a) \quad R = \begin{pmatrix} 0 \\ R_1 \\ R_2 \\ R_3 \end{pmatrix},$$

where R_i is any complex number $R_j = L_j + iM_j$. However, Ψ^A , as it stands, is not suitable for a final field because it is not purely transverse. Hence we shall subtract the part whose divergence is not zero.

In accordance with the previous sections we may expand $\Psi^A(\underline{x})$ as

$$(6.23) \quad \Psi^A(\underline{x}) = \sum_{\epsilon=\pm 1} \int \chi(\underline{x}|\underline{p}, \epsilon) \Psi^A(\underline{p}, \epsilon) d\underline{p} + \int \chi(\underline{x}|\underline{p}, 0) \Psi^A(\underline{p}, 0) d\underline{p}.$$

Now, the part for which the divergence does not vanish is

$$(6.24) \quad \Psi^{AL}(\underline{x}) = \int \chi(\underline{x}|\underline{p}, 0) \Psi^A(\underline{p}, 0) d\underline{p}.$$

Moreover, since

$$(6.25) \quad \Psi^A(\underline{p}, 0) = \sum_i \int \chi_i^*(\underline{x}|\underline{p}, 0) \Psi_i^A(\underline{x}) d\underline{x}$$

we have, on using the explicit form of $\chi_i(\underline{x}|\underline{p}, 0)$,

$$(6.26) \quad \begin{aligned} \Psi_i^{AL}(\underline{x}) &= \sum_j \int d\underline{x}' \Psi_j^A(\underline{x}') \int \chi_i(\underline{x}|\underline{p}, 0) \chi_j^*(\underline{x}'|\underline{p}, 0) d\underline{p} \\ &= -\frac{1}{4\pi} \sum_{j=1}^3 R_j \frac{\partial^2}{\partial x^i \partial x^j} \frac{1}{|\underline{x}|}, \quad (i=1,2,3). \end{aligned}$$

Hence, we shall take as the field immediately after the sources are switched off

$$(6.27) \quad \Psi_i = R_i \delta(\underline{x}) + \frac{1}{4\pi} \sum_{j=1}^3 R_j \frac{\partial^2}{\partial x^i \partial x^j} \frac{1}{|\underline{x}|}, \quad (i=1,2,3),$$

which is still highly localized near the origin and is now a purely transverse wave.

The sources which give rise to (6.27) are easily calculated:

$$(6.28) \quad \begin{cases} \Phi_0^E(\underline{x}) = 0 \\ \Phi_i^E(\underline{x}) = \frac{1}{4\pi A} \left[M_i \delta(\underline{x}) + \frac{1}{4\pi} \sum_{j=1}^3 M_j \frac{\partial^2}{\partial x^i \partial x^j} \frac{1}{|\underline{x}|} \right], \quad (i=1,2,3) \end{cases}$$

$$(6.28a) \quad \begin{cases} \Phi_0^U(\underline{x}) = 0 \\ \Phi_1^U(\underline{x}) = \frac{1}{(4\pi)^2_B} \left[L_3 \frac{\partial}{\partial x^2} - L_2 \frac{\partial}{\partial x^3} \right] \frac{1}{|\underline{x}|} \\ \Phi_2^U(\underline{x}) = \frac{1}{(4\pi)^2_B} \left[L_1 \frac{\partial}{\partial x^3} - L_3 \frac{\partial}{\partial x^1} \right] \frac{1}{|\underline{x}|} \\ \Phi_3^U(\underline{x}) = \frac{1}{(4\pi)^2_B} \left[L_2 \frac{\partial}{\partial x^1} - L_1 \frac{\partial}{\partial x^2} \right] \frac{1}{|\underline{x}|} . \end{cases}$$

In addition to these sources, one can add the arbitrary sources given by (6.19).

For $t > 0$, it is easy to show that the field is given by

$$(6.29) \quad \Psi_i(\underline{x};t) = \sum_j G_{ij}(\underline{x};t|0;0)R_j$$

where G is the Green's function given by (3.17) or (3.18).

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